Springer

KSME

Journal of Mechanical Science and Technology 24 (4) (2010) 899~908
www.springerlink.com/content/1738-494x
DOI 10.1007/s12206-010-0215-9

# Thermodynamic second law analysis for a gravity-driven variable viscosity liquid film along an inclined heated plate with convective cooling †

O. D. Makinde*

*Faculty of Engineering, Cape Peninsula University of Technology, P. O. Box 1906, Bellville 7535, South Africa*

(Manuscript Received June 8, 2009; Revised January 14, 2010; Accepted February 12, 2010)

---

Abstract

The inherent irreversibility and thermal stability in a gravity-driven temperature-dependent variable viscosity thin liquid film along an inclined heated plate with convective cooling is investigated. In this study, both the isothermal and isoflux heating of the plate are considered. The free surface of the liquid film is assumed to exchange heat with the surroundings following Newton's cooling law and the fluid viscosity model varies as an inverse linear function of the temperature. Analytical solutions are constructed for the governing boundary-value problem, and important properties of velocity and temperature fields such as thermal stability criterion are obtained. Expressions for volumetric entropy generation numbers, irreversibility distribution ratio, and the Bejan number in the flow field are also obtained and discussed quantitatively.

*Keywords*: Inclined plate; Liquid film; Variable viscosity; Thermal stability; Convective cooling; Entropy analysis

---

## 1. Introduction

The occurrence of thin liquid films flowing down from an inclined plate has been encountered in many experimental setups and technological applications. In the literature, the study of thin liquid films has a long history [1-3]. Apart from permitting a fundamental simplification of the viscous equations governing free surface flows, the tracking of such films has a significant impact on the manufacturing and final quality of the product [4]. Examples abound, such as casting processes, hot rolling, wire drawing, mold filling, thin film processes, extrusion, coatings, spray deposition, and fluid jetting devices in which material interfaces are present. Flows occurring on inclined plates have been numerically investigated in the work of Rahman et al. [5]. They studied the free surface of a thin liquid film in the presence and absence of gravitational body force using a boundary-fitted coordinate system where the irregular free surface conformed to one of the flow boundaries. Therien et al. [6] measured the film thickness for power law fluids flowing down inclined plates and compared the results with an analytical expression with good agreement. Sylvester et al. [7] compared experimental and predicted film thicknesses for power law fluids flowing down a vertical wall in the laminar and wavy fully developed regimes.

† This paper was recommended for publication in revised form by Associate Editor Tong Seop Kim

*Corresponding author. Tel.: +27 21 9596644, Fax.: +27 21 5550775
E-mail address: makinded@cput.ac.za

© KSME & Springer 2010

Analysis of thermal boundary layer flows of a thin liquid film on a heated inclined plate is extremely important in many industrial applications, especially in the area of handling and processing of such fluid. In the classical treatment of thermal boundary layers, the kinematic viscosity is assumed to be constant; however, experiments have indicated that this assumption only makes sense if the temperature does not change rapidly for the application of interest [4]. Indeed, for liquids, experimental data showed that viscosity decreased with temperature [3, 4, 8]. Makinde [9, 10] employed the Hermite-Padé approximation technique to investigate the problem of heat transfer in steady flows of a liquid film with adiabatic free surface along an inclined heat plate. The study was later extended to include heat transfer analysis in gravity-driven films of reactive non-Newtonian liquids [11].

Moreover, all real processes present irreversibilities and can be associated with friction and thermal gradients, among others, resulting in process efficiency loss. This efficiency loss is related to the entropy generation [12, 13]. As the generation of entropy destroys system energy, its minimization has been used as the optimal design criteria for thermal systems. The study of entropy generation in conductive and convective heat transfer processes has assumed considerable importance since the pioneering work of Bejan [14] and his subsequent book on the subject [15]. Many entropy generation studies on flows with heat transfer can be found in the literature [16-20]. Sahin [21] analyzed the variation of entropy generation in the function of viscosity in forced flow. Makinde [22] reported analytical solutions for entropy generation, irreversibility distribution and Bejan number for a variable-viscosity channel flow with non-uniform wall temperature, from the analytic solution of velocity and temperature profile. Recently, the thermodynamic second law characteristics for variable viscosity channel flow with convective cooling at the walls have also been discussed by Makinde [23].

In all of the abovementioned studies, the thermodynamic second law analysis for gravity-driven flow over both the isothermal- and isoflux-heated inclined plates with convective heat exchange at the free surface have not yet been investigated. Thus, in the present paper, we extend the work of Makinde [10] on variable viscosity thin liquid films with a free surface flowing down a slightly inclined smooth solid substrate to include entropy generation analysis and the effect of convective cooling at the free surface. The substrate is subjected to both isothermal and isoflux heating and the (infinitely extended) air layer above it is taken to be passive. Heat transfer from the film-free surface to the air is taken into account using Newton's law of cooling at the liquid-air interface; the film viscosity is then modeled as an inverse linear function of the temperature [24]. The plan of this paper is as follows: Sections 2 and 3 describe the theoretical analysis of the problem with respect to the fluid velocity and temperature fields; Section 4 describes the thermal stability criterion for the flow system; Section 5 presents and obtained the volumetric entropy generation rate, irreversibility distribution ratio, and the Bejan number; and Section 6 presents the graphic results and their discussion.

## 2. Mathematical model

We considered an isothermal- and isoflux-heated inclined plate surfaces placed in parallel streams of hydrodynamically- and thermally-developed liquid films (Fig. 1). It has been assumed that the fluid is Newtonian with temperature dependent viscosity and all other fluid properties remaining constant.

Neglecting the inertia terms, the momentum and energy balance equations take the form [2-5, 9, 10, 17]:

$$\frac{d^2T}{d\bar{y}^2}+\frac{\bar{\mu}}{k}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2=0,\quad \frac{d}{d\bar{y}}\left(\bar{\mu}\frac{d\bar{u}}{d\bar{y}}\right)+\rho g\sin(\phi)=0, \tag{1}$$

In order to investigate both the isothermal- and isoflux-heated inclined plate situations, we write the boundary conditions at the plate surface as:

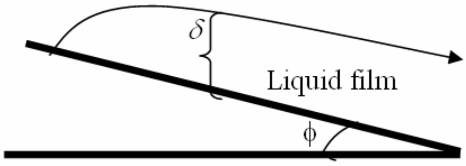


Fig. 1. Geometry of the problem.

$$\bar{u}=0\,,\quad ak\frac{dT}{d\bar{y}}-\frac{bk}{\delta}(T-T_a)=-\frac{k}{\delta}(T_0-T_a)\,,\quad \text{on}\quad \bar{y}=0\,, \tag{2}$$

where $a$ =0, $b$ =1 for isothermal heating and $a$=1, $b$=0 for isoflux heating. The convective heat exchange with the ambient at the free surface is given as:

$$\frac{d\bar{u}}{d\bar{y}}=0\,,\quad k\frac{dT}{d\bar{y}}=-h(T-T_a)\quad \text{on}\quad \bar{y}=\delta\,, \tag{3}$$

where $\bar{u}$ is the axial velocity, $T$ is the absolute temperature, $T_0$ is the plate reference temperature (i.e., uniform temperature for isothermal case or initial plate temperature for isoflux case), $T_a$ is the ambient temperature, $h$ is the heat transfer coefficient, $k$ is the thermal conductivity, $\delta$ is the liquid film thickness, $\phi$ is the inclination angle, $\rho$ is the fluid density, $g$ is the gravitational acceleration, $\bar{y}$ is the vertical distance, $a$ and $b$ are the inclined plate heating parameters. The temperature dependency of dynamic viscosity ($\bar{\mu}$) can be expressed as:

$$\bar{\mu}=\frac{\mu_0}{1+m(T-T_a)}\,, \tag{4}$$

where $\mu_0$ is the fluid viscosity at ambient temperature $T_a$ and the coefficient $m$ determines the strength of dependency between viscosity and temperature. Eq. (4) is most appropriate for liquid film since the viscosity of liquid decreases as the temperature increases [24]. The following dimensionless quantities are introduced into Eqs. (1)-(4):

$$\begin{gathered}\theta=\frac{T-T_a}{T_0-T_a},\ y=\frac{\bar{y}}{\delta},\ Br=\frac{\left(\delta^2\rho g\sin(\phi)\right)^2}{\mu_0 k(T_0-T_a)},\\ u=\frac{\mu_0\bar{u}}{\delta^2\rho g\sin(\phi)},\ \alpha=m(T_0-T_a),\ \mu=\frac{\bar{\mu}}{\mu_0},\ Bi=\frac{h\delta}{k}.\end{gathered} \tag{5}$$

The dimensionless governing equations combined with the corresponding boundary conditions given as:

$$\frac{d^2\theta}{dy^2}+\mu Br\left(\frac{du}{dy}\right)^2=0,\ \frac{d}{dy}\left(\mu\frac{du}{dy}\right)=-1\,, \tag{6}$$

with

$$u=0,\quad a\frac{d\theta}{dy}-b\theta=-1\quad \text{at}\quad y=0,\quad \text{and} \tag{7}$$

$$\frac{du}{dr}=0\,,\quad \frac{d\theta}{dy}=-Bi\theta\,,\quad \text{at}\quad y=1, \tag{8}$$

where $\mu=1/(1+\alpha\theta)$, $Br$ is the Brinkmann number, $Bi$ is the Biot number, and $\alpha$ is the variable viscosity parameter. Eqs. (6)-(8) are solved analytically in the following section.

## 3. Solution method

Eqs. (6)-(8) subject to the boundary conditions can be easily combined to give:

$$\frac{d^2\theta}{dy^2} + Br(1-y)^2(1+\alpha\theta) = 0\ , \quad \text{and} \tag{9}$$

$$\frac{du}{dy} = (1-y)(1+\alpha\theta)\ , \tag{10}$$

with

$$\frac{d\theta}{dy}(1) = -Bi\theta(1)\ ,\ \ a\frac{d\theta}{dy}(0) - b\theta(0) = -1\ ,\ u\,(0)=0. \tag{11}$$

Using the computer symbolic algebra package MAPLE [26], the exact solution to Eq. (9) with the corresponding boundary conditions is obtained and the fluid temperature profiles are given as:

$$\theta(y) = \text{hypergeom}\left(\left[\frac{1}{4}\right],\left[\frac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\,(-1+y)^2\right) A1\, e^{-\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}\,y\,(y-2)} + \text{hypergeom}\left(\left[\frac{3}{4}\right],\left[\frac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\,(-1+y)^2\right)(-1+y)\,A2\, e^{-\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}\,y\,(y-2)} - \frac{1}{\alpha}, \tag{12}$$

where hypergeom(⋯) denotes a generalized hypergeometric function and is defined in the appendix. The constant coefficients A1 and A2 are given as:

$$A1 = -\frac{\alpha\, A2\, e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}} - Bi}{\alpha\, Bi\, e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}}}\ , \tag{13}$$

$$\begin{aligned} A2 = \Bigg(Bi\Bigg(&I\,a\,\alpha\,\text{hypergeom}\left(\left[\tfrac{5}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} \\ &- I\,a\,\alpha\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} \\ &+ b\sqrt{\alpha}\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right) - b\sqrt{\alpha}\,e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}} \\ &- \alpha^{3/2}\,e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}}\Bigg)\Bigg) \Bigg/ \Bigg(e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}}\Bigg(I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{5}{4}\right],\right. \\ &\left.\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} - I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right],\right. \\ &\left. I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} + I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{7}{4}\right],\left[\tfrac{5}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br}\,Bi \\ &+ a\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)Bi \\ &- I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br}\,Bi \\ &+ b\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right) \\ &+ b\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)Bi\Bigg)\Bigg). \end{aligned} \tag{14}$$

For isoflux heating calculation, we simply replace $a$ = 1 and $b$ = 0 in Eqs. (12)-(14), while for isothermal heating calculation we replace $a$ = 0 and $b$=1 in Eqs. (12)-(14). The fluid velocity profile governed by Eq. (10) is solved numerically as the initial value problem using fourth order Runge-Kutta algorithm [26].

## 4. Thermal stability criterion

For the temperatures in the flow field to remain finite at any given value of $\alpha > 0$, the denominator of Eq. (12) should not vanish [3, 11, 25]. The imposition of this restriction leads to the following thermal stability criterion:

$$\begin{aligned} \Bigg(&I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{5}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} \\ &- I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br} \\ &+ I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{7}{4}\right],\left[\tfrac{5}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br}\,Bi \\ &+ a\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)Bi \\ &- I\,a\,\alpha^2\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)\sqrt{Br}\,Bi \\ &+ b\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{1}{4}\right],\left[\tfrac{1}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right) \\ &+ b\,\alpha^{3/2}\,\text{hypergeom}\left(\left[\tfrac{3}{4}\right],\left[\tfrac{3}{2}\right], I\sqrt{Br}\sqrt{\alpha}\right)Bi\Bigg) \\ &Bi\,e^{\frac{1}{2}I\sqrt{Br}\sqrt{\alpha}}\,\alpha \neq 0. \end{aligned} \tag{15}$$

Eq. (15) indicates that the thermal stability of the flow system depends not only on the Biot number ($Bi$) but also on the Brinkmann number ($Br$), isothermal/isoflux heating parameters ($a$, $b$), as well as on the parameter characterizing the fluid viscosity variation ($\alpha$).

## 5. Entropy analysis

The general equation for the entropy generation per unit volume is given by [12-23];

$$S''' = \frac{k}{T_0^2}(\nabla T)^2 + \frac{\mu}{T_0}\Phi\ . \tag{16}$$

The first term in Eq. (16) refers to the irreversibility due to heat transfer, while the second term is the entropy generation due to viscous dissipation. Using Eq. (16), we express the entropy generation number in dimensionless form as:

$$Ns = \frac{\delta^2 T_0^2 S'''}{k(T_0 - T_a)^2} = \left(\frac{d\theta}{dy}\right)^2 + \frac{\mu Br}{\Omega}\left(\frac{du}{dy}\right)^2, \tag{17}$$

where $\Omega = (T_0 - T_a)/T_0$ is the temperature difference parameter. In Eq. (17), the first term can be assigned as $N_1$, while the second term due to viscous dissipation as $N_2$:

$$N_1 = \left(\frac{d\theta}{dy}\right)^2, \quad N_2 = \frac{\mu Br}{\Omega}\left(\frac{du}{dy}\right)^2. \tag{18}$$

To obtain an idea on whether fluid friction dominates over heat transfer irreversibility or vice-versa, Bejan [14, 15] defined the irreversibility distribution ratio as $\Phi = N_2/N_1$. Heat transfer dominates for $0 \le \Phi < 1$ and fluid friction dominates when $\Phi > 1$. The contribution of both heat transfer and fluid friction to entropy generation are equal when $\Phi = 1$. In many engineering designs and energy optimisation problems, the contribution of heat transfer entropy $N_1$ to overall entropy generation rate $Ns$ is required. As an alternative to the irreversibility parameter, the Bejan number ($Be$) is defined mathematically as:

$$Be = \frac{N_1}{Ns} = \frac{1}{1+\Phi} \, . \tag{19}$$

Clearly, the Bejan number ranges from 0 to 1, in which $Be = 0$ is the limit where the irreversibility is dominated by fluid friction effects, while $Be = 1$ corresponds to the limit where the irreversibility due to heat transfer by virtue of finite temperature differences dominates. The contribution of both heat transfer and fluid friction to entropy generation are equal when $Be = 1/2$.

## 6. Results and discussion

In this section, physically meaningful values of the various parameters embedded in the problem were utilized to validate our results numerically. The mathematical analysis in this investigation is valid for liquid whose viscosity decreases with an increase in temperature. It is very important to note that in isoflux-heating situations, constant heat transfer rate is maintained at the plate surface whereas in isothermal heating the plate surface is maintained at uniform temperature. Furthermore, a positive increase in the parameter value of $\alpha$ indicated a decrease in fluid viscosity, while the convective cooling in the flow system was enhanced by increasing the Biot number ($Bi$). We also observed that the Brinkmann number ($Br$) depended on the angle of inclination ($\phi$) to the plane (see Eq. 5), and showed the effect of viscous heating on the flow system. As the inclination angle increased, $Br$ also increased.

Table 1 shows the thermal criticality values for the Brinkmann number ($Br_c$) in the isothermal- heated inclined plate surface flow system. The computation of $Br_c$ was based on the thermal stability criterion highlighted in Eq. (15). This criterion is extremely important from an engineering point of view for it provides the necessary safety conditions for this material during processing and handling to avoid the thermal runaway scenario. For $0 \le Br < Br_c$, the flow was thermally stable. However at $Br_c$, the fluid temperature was no longer finite and the flow became thermally unstable, displaying a classical form indicating thermal runaway. It is very interesting to note that the magnitude $Br_c$ increased with increasing value of $Bi$ and decreased with increasing value of α. This implies that an increase in convective cooling will augment the thermal stability in the flow system, while a decrease in the fluid viscosity will diminish it, thus leading to the early development of thermal runaway and thermal instability in the flow system. The case of uniform heat flux (isoflux) is illustrated in Table 2. Although the magnitude of thermal criticality values is lower than that of the isothermal case, the trend is similar (i.e., that the magnitude $Br_c$ also decreased with increasing value of α and increased with increasing value of $Bi$). From the two tables, it is noteworthy that the liquid flow over the isothermal-heated inclined plate surface is more thermally stable than the flow over the isoflux-heated inclined plate surface.

Table 1. Computations showing thermal criticality values for the isothermal heating case a=0, b=1.

| α | Bi | $Br_c$ |
|---|---|---|
| 1.0 | 0.1 | 16.88320906 |
| 3.0 | 0.1 | 5.627736355 |
| 5.0 | 0.1 | 3.376641813 |
| 7.0 | 0.1 | 2.411887009 |
| 9.0 | 0.1 | 1.875912118 |
| 1.0 | 0.3 | 18.23498521 |
| 1.0 | 0.5 | 19.35958752 |
| 1.0 | 0.7 | 20.30794104 |
| 1.0 | 0.9 | 21.11726352 |
| 1.0 | 1.0 | 21.47873342 |

Table 2. Computations showing thermal criticality values for the isoflux heating case a=1, b=0.

| α | Bi | $Br_c$ |
|---|---|---|
| 1.0 | 0.5 | 1.132445757 |
| 3.0 | 0.5 | 0.3774819190 |
| 5.0 | 0.5 | 0.2264891514 |
| 7.0 | 0.5 | 0.1617779653 |
| 9.0 | 0.5 | 0.1258273063 |
| 1.0 | 0.7 | 1.442674351 |

### *6.1 Effects of parameter variation on the velocity profiles*

In Figs. 2-7, the typical variations of the fluid velocity profiles in the normal direction are depicted for both the isothermal- and isoflux-heated inclined plate surfaces. Generally, the velocity profiles showed a transverse increase from the inclined plate surface and attained its maximum value at the free surface. For the isothermal heating case, the velocity increased with increasing values of $\alpha$, $Br$ and decreased with increasing values of $Bi$. A similar trend was observed for the isoflux-heated inclined plate surface. Thus, a decrease in fluid viscosity and an increase in viscous heating will enhance flow velocity, while an increase in convective cooling will slow down the flow process. Moreover, it is noteworthy that the magnitude of fluid velocity under the isoflux heating condition is higher than that of the isothermal case.

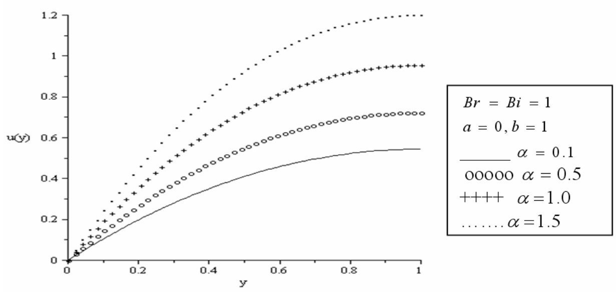


Fig. 2. Effects of viscosity variation on the velocity profiles for the isothermal case.

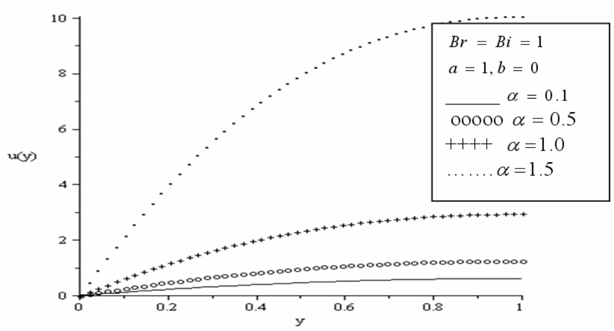


Fig. 3. Effects of viscosity variation on the velocity profiles for the isoflux case.

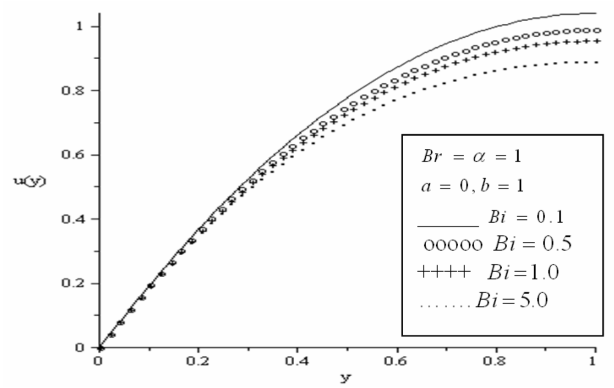


Fig. 4. Effects of convective cooling on the velocity profiles for the isothermal case.

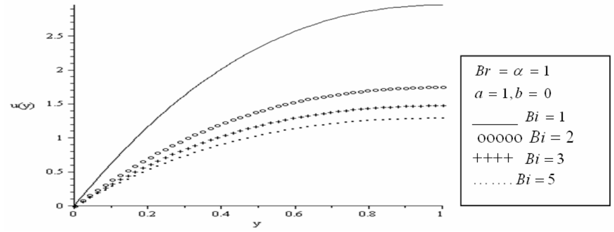


Fig. 5. Effects of convective cooling on the velocity profiles for the isoflux case.

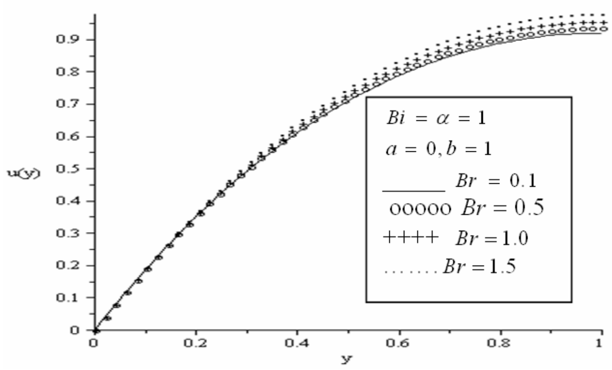


Fig. 6. Effects of the Brinkman number on the velocity profiles for the isothermal case.

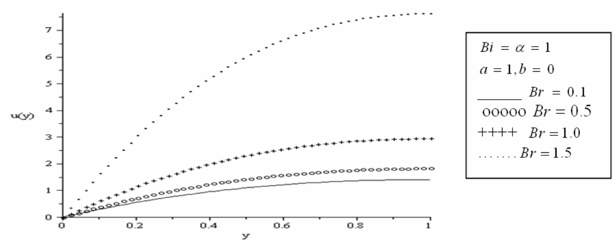


Fig. 7. Effects of the Brinkman number on the velocity profiles for the isoflux case.

### *6.2 Effects of parameter variation on the temperature profiles*

The fluid temperature profiles are depicted in Figs. 8-13. For the case of the isothermal-heated inclined plate, it is shown that the fluid temperature is united at the plate surface satisfying the boundary condition, and then decreased gradually towards the liquid-free surface due to convective heat exchange with the ambient (Figs. 8, 10 and 12). As can be seen from Figs. 9, 11 and 13, the trend is different for the isoflux-heated inclined plate surface. Here, we observed that the fluid temperature at the plate surface is not uniform; however, the peak value for the fluid temperature is attained at the plate surface and then decreased transversely towards the liquid free surface due to convective cooling. Moreover, it is also noteworthy that the fluid temperature generally increased with increasing values of $\alpha$, $Br$ and decreased with increasing values of the Biot number $Bi$ in both the isothermal- and isoflux-heated inclined plate surfaces. Therefore, a decrease in fluid viscosity and an increase in viscous heating will augment fluid temperature, while an increase in the convective heat exchange with the cool environment at the liquid-free surface will reduce fluid temperature.

### *6.3 Effects of parameter variation on the entropy generation rate*

The numerical results for the entropy generation rate are displayed in Figs. 14-19, which show that the entropy generation rate is minimum at the plate surface and then increased to its peak value at the liquid free surface for the isoflux-heated

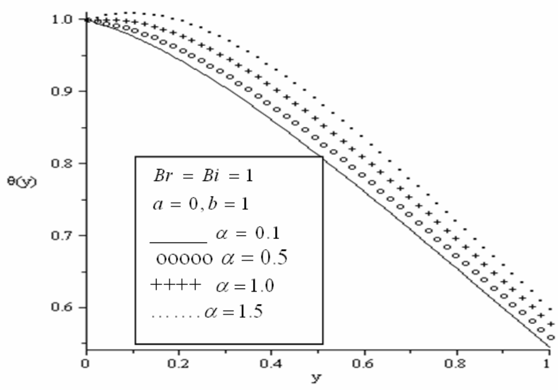


Fig. 8. Effects of viscosity variation on the temperature profiles for the isothermal case.

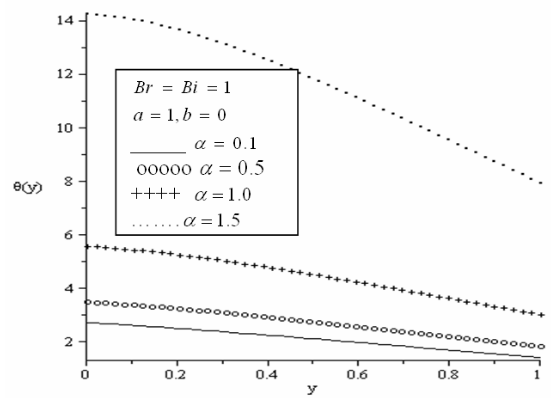


Fig. 9. Effects of viscosity variation on the temperature profiles for the isoflux case.

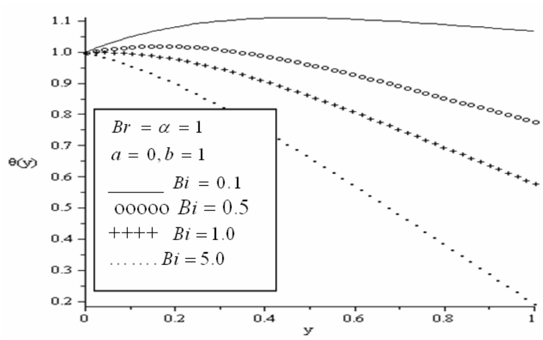


Fig. 10. Effects of convective cooling on the temperature profiles for the isothermal case.

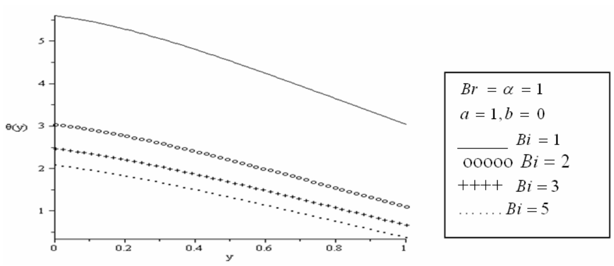


Fig. 11. Effects of convective cooling on the temperature profiles for the isoflux case.

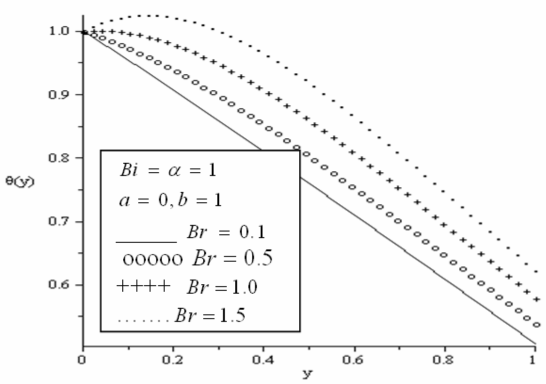


Fig. 12. Effects of the Brinkman number on the temperature profiles for the isothermal case.

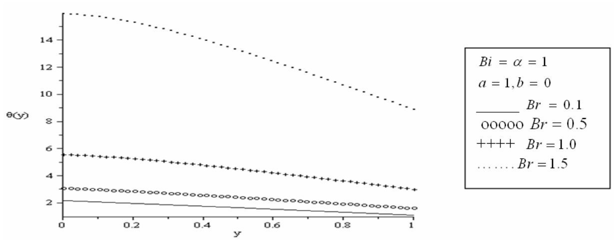


Fig. 13. Effects of the Brinkman number on the temperature profiles for the isoflux case.

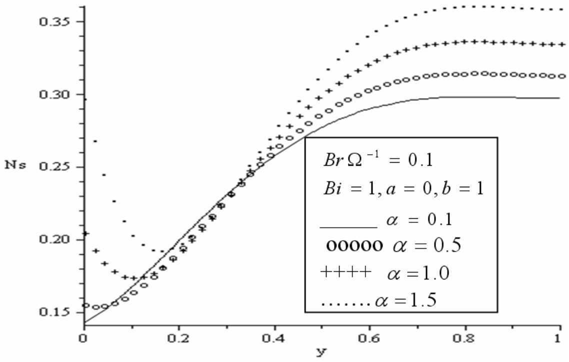


Fig. 14. Effects of viscosity variation on the entropy generation rate for the isothermal case.

inclined plate. Entropy generated in the flow system increased with increasing values of $\alpha$, $Br\Omega^{-1}$ (Figs. 15 and 17); however, it decreased with increasing values of *Bi* (Fig. 19). This implies that for the isoflux-heated inclined plate surface, a decrease in fluid viscosity and an increase in viscous heating represented by the group parameter ($Br\Omega^{-1}$) will enhance the entropy generation rate, while an increase in the convective cooling will diminish entropy generation rate. The trend is different for the isothermal-heated inclined plate. The entropy generation rate increased at both the plate surface and the liquid free surface increased with increasing values of $\alpha$, $Br\Omega^{-1}$ and *Bi* as well (Figs. 14, 16 and 18). Thus, a decrease in fluid viscosity and an increase in viscous heating and convective cooling will enhance entropy generation rate in a flow system with an isothermal-heated inclined plate surface.

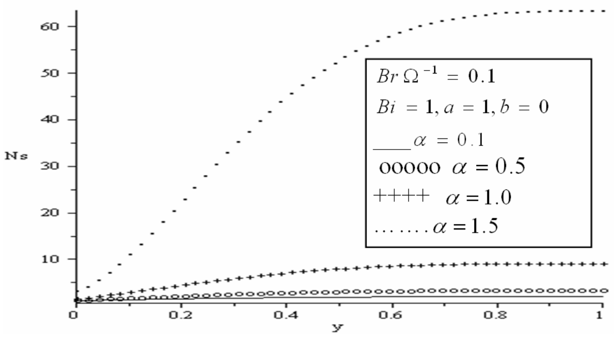


Fig. 15. Effects of viscosity variation on the entropy generation rate for the isoflux case.

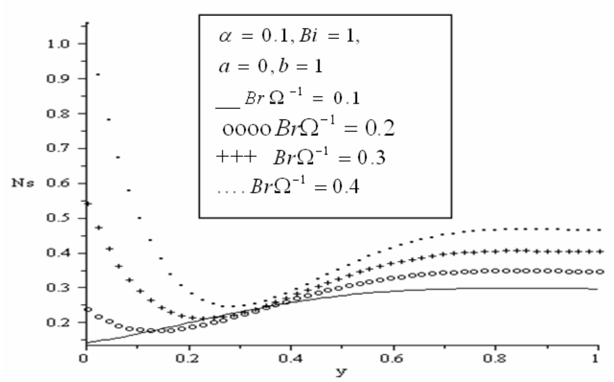


Fig. 16. Effects of group parameter variation on the entropy generation rate for the isothermal case.

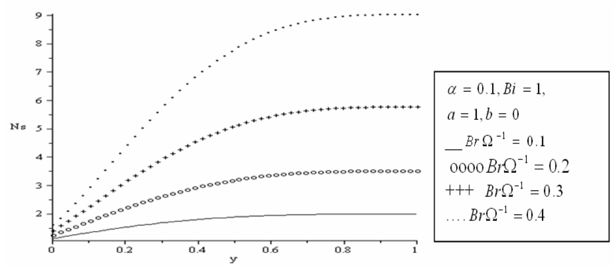


Fig. 17. Effects of group parameter variation on the entropy generation rate for the isoflux case.

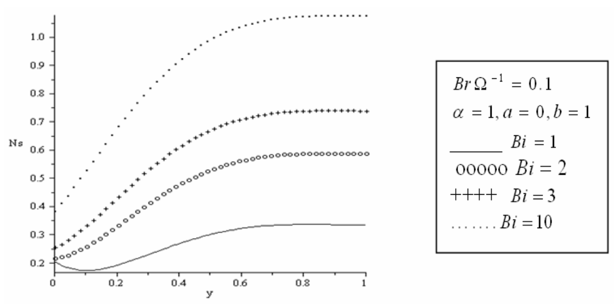


Fig. 18. Effects of convective cooling on the entropy generation rate for the isothermal case.

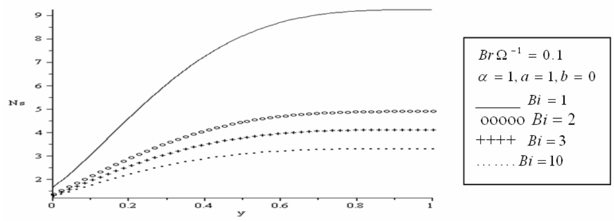


Fig. 19. Effects of convective cooling on the entropy generation rate for the isoflux case.

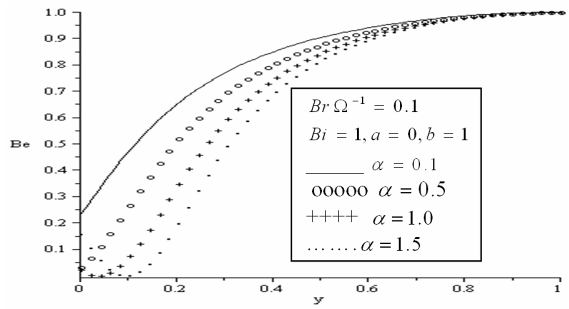


Fig. 20. Effects of viscosity variation on the Bejan number for the isothermal case.

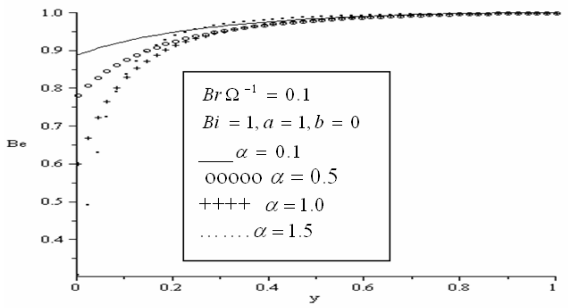


Fig. 21. Effects of viscosity variation on the Bejan number for the isoflux case.

### *6.4 Effects of parameter variation on the Bejan number*

In Figs. 20-25, the Bejan (*Be*) number is illustrated for various parametric values. It is interesting to note that the fluid friction irreversibility dominated the flow system at the inclined heated plate surface, while the heat transfer irreversibility strongly dominated at the liquid free surface. Generally, the dominance effect of fluid friction irreversibility at the isothermal-heated inclined plate surface is more pronounced than that in the isoflux-heated inclined plate surface. The Bejan number at the heated inclined plate surface decreased with increasing values of $\alpha$ and $Br\Omega^{-1}$ (Figs. 20-23). Thus, a decrease in fluid viscosity and an increase in viscous heating will enhance the dominance effect of fluid friction irreversibility at the heated inclined plate surface. Moreover, an increase in the convective heat exchange at the liquid-air interface will enhance the dominance effect of heat transfer irreversibility at the liquid free surface, but diminish the dominance effect of fluid friction irreversibility at the inclined heated plate surface (Figs. 24 and 25).

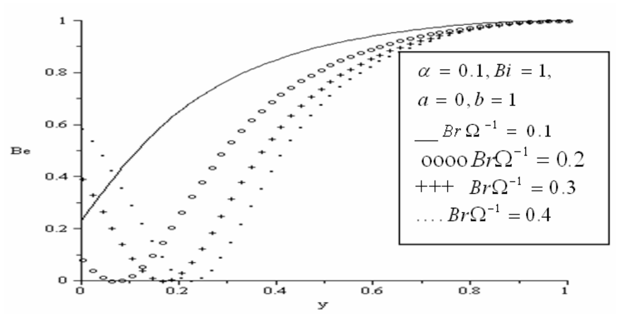


Fig. 22. Effects of group parameter variation on the Bejan number for the isothermal case.

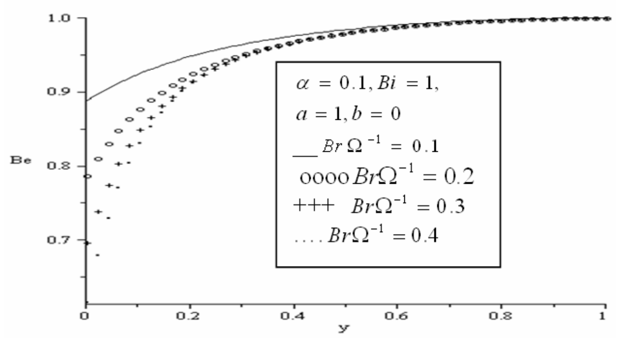


Fig. 23. Effects of group parameter variation on the Bejan number for the isoflux case.

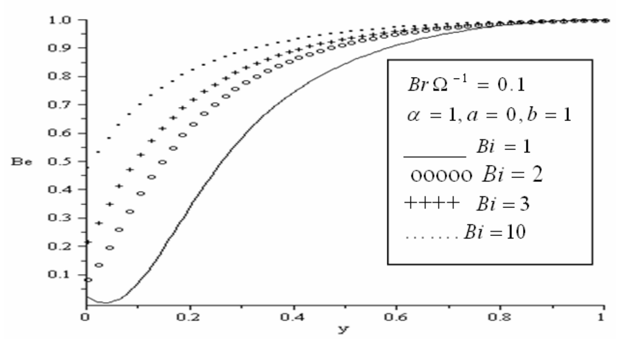


Fig. 24. Effects of convective cooling on the Bejan number for the isothermal case.

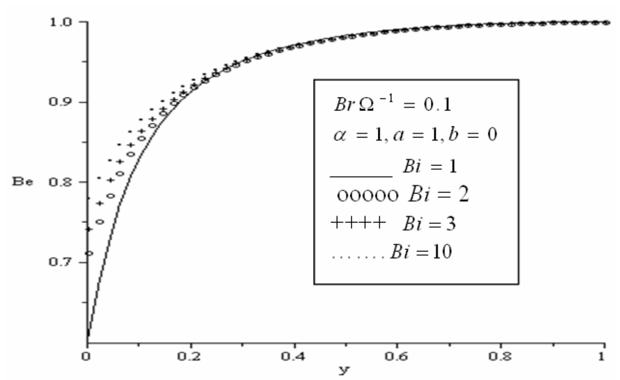


Fig. 25. Effects of convective cooling on the Bejan number for the isoflux case.

## 7. Conclusion

The entropy production rates for variable viscosity liquid film along both isothermal- and isoflux-heated inclined plate surfaces with convective cooling at the liquid free surface was investigated. Both analytical and numerical techniques were employed to obtain the flow velocity and temperature profiles. The expressions for thermal stability criterion, volumetric entropy generation rate, and Bejan number were obtained. Our results revealed that both the fluid velocity and temperature decreased with increasing convective cooling ($Bi$) and increased with increasing values of $\alpha$, $Br$. The flow system over an isothermal-heated inclined plate surface is more thermally stable than the case of the isoflux-heated plate surface. Moreover, viscous dissipation irreversibility dominated the flow system at the inclined heated plate surface while the heat transfer irreversibility strongly dominated at the liquid-free surface. The dominance effect of fluid friction irreversibility at the isothermal-heated inclined plate surface is more pronounced compared with the isoflux-heated inclined plate surface. Since the performance and optimum design of any thermal system can be enhanced by the ability to clearly identify the source and location of entropy, the present results show that a minimum entropy generation system can be designed with the proper choice and appropriate combination of the various thermophysical parameters.

## Acknowledgements

The author would like to thank the National Research Foundation (NRF) of South Africa Thuthuka programme for their generous financial support. The suggestions for improvement from the anonymous referees are also highly appreciated.

## Nomenclature

T : Fluid temperature
$T_0$ : Plate reference temperature
$\bar{y}$ : Normal distance
$k$ : Thermal conductivity
$h$ : Heat transfer coefficient
$\bar{u}$ : Axial velocity component
$T_a$ : Ambient temperature
g : Gravitational acceleration
$Br$ : Brinkmann number
$Bi$ : Biot number
$Be$ : Bejan number
$m$ : Variable viscosity parameter
$u$ : Dimensionless axial velocity
$a$ : Isoflux heating parameters
$b$ : Isothermal heating parameter
$Ns$ : Entropy generation number
$y$ : Dimensionless normal distance

### *Greek symbols*

$\phi$ : Inclination angle

$\delta$ : Liquid film thickness
$\theta$ : Dimensionless temperature
$\mu$ : Fluid dynamic viscosity
$\alpha$ : Variable viscosity parameter
$\rho$ : Fluid density
$\mu_0$ : Viscosity at ambient temperature
$\Phi$ : Irreversibility distribution ratio
$\Omega$ : Temperature difference parameter
$\bar{\mu}$ : Fluid dynamic viscosity

## References


[1] H. I. Anderson, The momentum integral approach to laminar thin film flow, Proc. ASME Symp. on Thin Fluid Films, *Cincinatti, OH, FED*., 48 (1987) 7-13.

[2] G. Astarita, G. Marrucci and G. Palumbo, Non-Newtonian gravity flow along inclined plane surfaces, *Ind. Eng. Chem. Fundam*., 3 (4) (1964) 333-339.

[3] T. Cebeci, and P. Bradshaw, *Physical and computational aspects of convective heat transfer*, Springer-Verlag, New York, USA, (1984).

[4] H. Schlichting, *Boundary layer theory*, Springer-Verlag, New York, USA, (2000).

[5] M. M. Rahman, A. Faghri, W. L. Hankey and T. D. Swanson, Computation of the free surface flow of a thin liquid film at zero and normal gravity, *Numer.Heat Transf., Part A Appl.,* 17 (1) (1990) 53-71.

[6] N. Therien, B. Coupal and J. L. Corneille, Verification experimentale de l'epaisseur du film pour des liquides non-Newtonien s'ecoulant pargravite sur un plan incline, *Can. J. Chem. Eng.,* 48 (1970) 17-20.

[7] N. D. Sylvester, J. S. Tyler and A. H. P. Skelland, Non-Newtonian thin films: theory and experiment, *Can. J. Chem. Eng.,* 51 (1973) 418-429.

[8] O. D. Makinde, Heat and mass transfer in a pipe with moving surface-effect of viscosity variation and energy dissipation, *Quaestiones Mathematicae,* 24 (2001) 93-104.

[9] O. D. Makinde, Hermite-Padé approximation approach to steady flow of a liquid film with adiabatic free surface along an inclined heat plate, *Physica A,* 381 (2007) 1-7.

[10] O. D. Makinde, Laminar falling liquid film with variable viscosity along an inclined heated plate, *Applied Mathematics and Computation,* 175 (2006) 80-88.

[11] O. D. Makinde, Thermal criticality for a reactive gravity driven thin film flow of a third grade fluid with adiabatic free surface down an inclined plane, *Applied Mathematics and Mechanics,* 30 (3) (2009) 373-380.

[12] U. Narusawa, The second law analysis of mixed convection in rectangular ducts, *Heat and Mass Transfer,* 37 (2001) 197-203.

[13] G. Ibanez, S. Cuevas and M. Lopez de Haro, Minimization of entropy generation by asymmetric convective cooling, *Int. J. Heat Mass Transfer,* 46 (2003) 1321-1328.

[14] A. Bejan, *Entropy generation through heat and fluid Flow*, John Wiley & Sons. Inc.: Canada, Chapter 5, (1994) 98.

[15] A. Bejan, *Entropy generation minimization*, CRC Press, Boca Raton, Florida, (1996).

[16] O. D. Makinde, Irreversibility analysis for gravity driven non-Newtonian liquid film along an inclined isothermal plate, *Physica Scripta,* 74 (2006) 642-645.

[17] S. Saouli and S. Aiboud-Saouli, Second law analysis of laminar falling liquid film along an inclined heated plate, *Inter. Comm. Heat Mass Transfer,* 31 (2004) 879-886.

[18] R. S. Reddy Gorla, L. W. Byrd and D. M. Pratt, Second law analysis for microscale flow and heat transfer, *Applied Thermal Engineering,* 27 (2007) 1414-1423.

[19] B. N. Taufiq, H. H. Masjuki, T. M. I. Mahlia, R. Saidur, M. S. Faizul and E. Niza Mohamad, Second law analysis for optimal thermal design of radial fin geometry by convection, *Applied Thermal Engineering,* 27 (2007) 1363-1370.

[20] S. H. Tasnim and S. Mahmud, Entropy generation in a vertical concentric channel with temperature dependent viscosity, *Int. Comm. Heat Mass Transfer,* 29 (7) (2002) 907-918.

[21] A. Z. Sahin, Effect of variable viscosity on the entropy generation and pumping power in a laminar fluid flow through a duct subjected to constant heat flux, *Heat Mass Transfer,* 35 (1999) 499-506.

[22] O. D. Makinde, Entropy-generation analysis for variable-viscosity channel flow with non-uniform wall temperature, *Applied Energy,* 85 (2008) 384-393.

[23] O. D. Makinde, Irreversibility analysis of variable viscosity channel flow with convective cooling at the walls, *Canadian Journal of Physics,* 86 (2) (2008) 383-389.

[24] E. M. A. Elbashbeshy and M. A. A. Bazid, The effect of temperature dependent viscosity on heat transfer over a continuous moving surface, *Journal of Applied Phys*., 33 (2000) 2716-2721.

[25] W. Squire, *A mathematical analysis of self-ignition*, *Applications of Undergraduate Mathematics in Engineering,* Ed. Noble, B. MacMillan, New York, USA, (1967).

[26] http://maplesoft.com/products/maple/technical.aspx


## Appendix

The MAPLE software utilized hypergeom ($a$, $b$; $c$; $x$) to represent generalized hypergeometric function. It is a special function and the solution to the hypergeometric differential equation is represented as:

$$x(1-x)y'' + \left\{c-(a+b+1)x\right\}y' - aby = 0 \ .$$

This differential equation has a regular singular point at the origin and the solution is given in series form as:

$$hypergeom(a,b;c;x) = 1 + \frac{ab}{1!c}x + \frac{a(a+1)b(b+1)}{2!c(c+1)}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}x^3 + \ldots$$

**Oluwole Daniel Makinde** is currently Senior Professor and Chair of Postgraduate Studies in the Faculty of Engineering Cape-Peninsula University of Technology (CPUT) in South Africa. Prior to joining CPUT, he was a Professor and Head of Applied Mathematics Department for many years at University of Limpopo, South Africa. He received his B.Sc. (Hons) First Class and M.Sc. from Obafemi Awolowo University, Ile-Ife, Nigeria and PhD at University Bristol, England all in Computational and Applied Mathematics. Professor Makinde has taught supervised graduate students and served as an external examiner at many universities within and outside Africa. He has co-authored two applied mathematics textbooks, two Open University monographs on Particle Mechanics and Incompressible Flow Theory and published numerous research articles in several reputable international journals. He is presently the Secretary General of African Mathematical Union. Professor Makinde received Best Senior Research Scientists Award at University of Limpopo African Mathematician Medal for Excellence in Applied Mathematics and many other awards and grants.